Lecture 6

Wednesday, January 29, 2020 5:44 AM

__Recall:__ Hartogs-II : Let $\Omega \subseteq \mathbb{C}^n$, $K \subset\subset \Omega$ s.t. $\Omega \setminus K$ is connected. If u is holom. in $\Omega \setminus K$, then $\exists\, U$ holom. in Ω s.t. $U|_{\Omega\setminus K} = u$.

__Brief review of pf:__

Let $\varphi \in C_0^\infty(\Omega)$ s.t. $\varphi \equiv 1$ on K, $u_0 = (1-\varphi)u$, and we look for $U = u_0 + v$. We set $f = -\bar\partial u_0$ and extend by 0 outside Ω to $f \in C^\infty_{0,(0,1)}(\Omega)$ w/ support in $\operatorname{supp}\varphi \subset\subset \Omega \subseteq \mathbb{C}^n$. Since $\bar\partial f = 0$, we can solve $\bar\partial v = f$ for $v \in C_0^\infty(\mathbb{C}^n)$ and $v \equiv 0$ in unbdd component G_∞ of $\mathbb{C}^n \setminus \operatorname{supp}\varphi$.

__Ex__: If $K' \subset\subset \Omega$ (compact), then for every component G of $\mathbb{C}^n \setminus K'$, we have $\Omega \cap G \neq \emptyset$.

Ex $\Rightarrow$ $G_\infty \cap \Omega$ is a nonempty open subset of $\Omega \setminus K$ (since $K \subseteq \operatorname{supp}\varphi$). Thus, $U = u$ on open subset of $\Omega \setminus K$, which is connected $\Rightarrow U = u$ on $\Omega \setminus K$ as desired.

__Prop 1.__ Let u be holom. in a polydisk $D^n \subseteq \mathbb{C}^n$ $\boxed{n>1}$ and let $Z_u := \{z \in D^n : u(z) = 0\}$. If $Z_u \neq \emptyset$, then Z_u is __not__ compact in D^n.

__Rem.__ Clearly not true in a disk $D \subseteq \mathbb{C}$.

__Pf.__ Suppose $Z_u \neq \emptyset$, $Z_u \subset\subset D^n$ compact. Then $\exists$ smaller polydisk $E^n = E_1 \times \dots \times E_n$, $\overline{E_j} \subseteq D_j$, such that $Z_u \subseteq E^n$. But $K = \overline{E^n}$ is compact in D^n, $D^n \setminus K$ connected, and $v = \frac{1}{u}$ is holomorphic in $D^n \setminus K$. By Hartogs-II, v is holomorphic in D^n, i.e. across Z_u, which is easily seen to be impossible. ☒

Boundary version of Hartogs – II. $\boxed{n>1}$

Bochner's Thm. Let $\Omega \subseteq \mathbb{C}^n$, be bdd, $\mathbb{C}^n \setminus \overline{\Omega}$ connected, and $\partial\Omega$ smooth, i.e. $\exists \rho \in \mathcal{C}^\infty(\mathbb{C}^n, \mathbb{R})$ s.t. $\partial\Omega = \{z \in \mathbb{C}^n : \rho(z) = 0\}$, $d\rho \neq 0$ on $\partial\Omega$ (a defining function for $\partial\Omega$). If $u \in \mathcal{C}^\infty(\overline{\Omega})$ and $\bar\partial u \wedge \bar\partial \rho = 0$ on $\partial\Omega$ ($u|_{\partial\Omega}$ is CR), then $\exists\, U \in \mathcal{C}^\infty(\overline{\Omega})$ s.t. U holom. in Ω and $U|_{\partial\Omega} = u$.

Rem. ① If such U exists, $U - u = a\rho \Rightarrow \bar\partial U - \bar\partial u = a\bar\partial\rho$ on $\partial\Omega = \{\rho = 0\}$. Thus, since U holom., $\bar\partial u \wedge \bar\partial\rho = a\bar\partial\rho \wedge \bar\partial\rho = 0$.

② **Ex.** $\bar\partial u \wedge \bar\partial\rho = 0$ on $\partial\Omega$ can be reformulated:

(CR) $\forall \zeta \in \partial\Omega$: $\displaystyle\sum_{j=1}^n c_j \frac{\partial u}{\partial \bar z_j}(\zeta) = 0$, $\forall c = (c_1, \dots, c_n)$ s.t. $\displaystyle\sum_{j=1}^n \frac{\partial\rho}{\partial \bar z_j}(\zeta) c_j = 0$

This is a condition that depends only on $u|_{\partial\Omega}$! Functions $u \in \mathcal{C}^1(\partial\Omega)$ that satisfy (CR) are called CR functions, and (CR) are called the tangential Cauchy-Riemann equations.

Pf of Bochner. Idea of pf is similar to Hartogs. Want to construct $U = u + v$ $\Rightarrow \bar\partial v = f = -\bar\partial u$, but now $\bar\partial u$ does not have compact support in Ω. Now, $\overline{\Omega}$ is compact in $\mathbb{C}^n$, but extending f by 0 in $\mathbb{C}^n \setminus \overline{\Omega}$ is not even continuous. We shall here prove existence of $U \in \mathcal{C}^1(\overline{\Omega})$ only.

Step 1. Find $U_0 \in \mathcal{C}^2(\overline{\Omega})$, $U_0|_{\partial\Omega} = u$, $\bar\partial U_0 = O(\rho^2)$ (vanishes to 2^{nd} order on $\partial\Omega$). By assumption, $\bar\partial u = h_0 \bar\partial\rho + h_1\rho$, $h_0 \in \mathcal{C}^\infty(\overline{\Omega})$, $h_1 \in \mathcal{C}^\infty_{(0,1)}(\overline{\Omega})$.

$$\Rightarrow \bar\partial(u - h_0\rho) = \bar\partial u - h_0\bar\partial\rho - \bar\partial h_0 \rho = \underbrace{(h_1 - \bar\partial h_0)}_{h_2}\rho = h_2\rho, \quad h_2 \in \mathcal{C}^\infty_{(0,1)}(\overline{\Omega})$$

Applying $\bar\partial$ again $\Rightarrow 0 = \bar\partial(h_2\rho) = \rho\bar\partial h_2 + h_2 \wedge \bar\partial\rho \Rightarrow h_2 \wedge \bar\partial\rho = 0$ on $\partial\Omega$.

$\Rightarrow h_2 = h_3\bar\partial\rho + h_4\rho$, $h_3 \in \mathcal{C}^\infty(\overline{\Omega})$, $h_4 \in \mathcal{C}^\infty_{(0,1)}(\overline{\Omega})$.

$\Rightarrow\ h_2 \overset{\cdot}{\overset{\cup}{=}} h_3 \bar\partial\rho + h_4\rho$, $h_3 \in \mathcal{C}^\infty(\bar\Omega)$, $h_4 \in \mathcal{C}^\infty_{(0,1)}(\bar\Omega)$.

Consider $U_0 = u - h_0\rho - h_3\frac{\rho^2}{2}$. Clearly, $U_0|_{\partial\Omega} = u$. And

$$\bar\partial U_0 = \bar\partial(u - h_0\rho) - h_3\rho\,\bar\partial\rho - \frac{\rho^2}{2}\bar\partial h_3 = \rho\underbrace{(h_2 - h_3\bar\partial\rho)}_{h_4\rho} - \frac{\rho^2}{2}\bar\partial h_3$$

$$= O(\rho^2).$$

<u>Step 2</u>. Def $f \in \mathcal{C}^1_{0,(0,1)}(\mathbb{C}^n)$ by $f = -\bar\partial U_0$ in $\bar\Omega$ and 0 in $\mathbb{C}^n \setminus \bar\Omega$.
The fact that f is $\mathcal{C}^1$ follows from $\bar\partial U_0 = O(\rho^2)$! Again $\bar\partial f = 0$.
Thus, by Thm on $\bar\partial$-eq. w/ compact supp, $\exists\ v \in \mathcal{C}^1_0(\mathbb{C}^n)$, $v \equiv 0$ in unbdd component G_∞ of $\mathbb{C}^n \setminus \bar\Omega$, s.t. $\bar\partial v = f$.
Consider $U = U_0 + v$. Then, $U \in \mathcal{C}^1(\bar\Omega)$, holom. in Ω since $\bar\partial U = \bar\partial U_0 + \bar\partial v = 0$ by construction of v. Moreover, since $v \equiv 0$ in G_∞ and $\mathbb{C}^n \setminus \bar\Omega = G_\infty$ (since $\mathbb{C}^n \setminus \bar\Omega$ is connected), $v = 0$ on $\partial\Omega \Rightarrow U|_{\partial\Omega} = U_0|_{\partial\Omega} = u$.

This proves existence of $U \in \mathcal{C}^1(\bar\Omega)$. To get $U \in \mathcal{C}^k(\bar\Omega)$, $k \geq 2$, we repeat step 1 to get U_0 w/ $\bar\partial U_0 = O(\rho^{k+1})$. For $\mathcal{C}^\infty$, this is more technical, but possible. ∎